Coalgebras for modelling behaviour

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The aim of programing is to force the computer to execute some actions and to generate a desired behaviour;

- the most important concept is a state - an abstraction of computer memory;
- execution of a program means a change of states;
- states are often hidden from observer;
- the aim of behavioural theory is to determine a relation between internal states and observable values;
- as a formal model coalgebras are used to provide observable behaviour of program systems;
A coalgebra is defined as a mapping

\[ c : \text{State} \to Q(\text{State}) ; \]

where

- \textbf{State} is the representation of states, state space;
- \( Q : C_{\text{State}} \to C_{\text{State}} \) is a polynomial endofunctor over a category of state representations.

To construct a coalgebra of a system

- we start with a signature \textit{State} of a state space specifying types and operations;
- we construct a base category \( C_{\text{State}} \) of state representations from a set \textit{State}, where morphisms are transitions (state changes);
- we construct a polynomial endofunctor \( Q \) over a category indicated by a given signature;
- we define a coalgebra \( c : \text{State} \to Q(\text{State}) \).
We introduce for \( J \) the following syntactic domains:

- \( n \in \text{Num} \) for digit strings;
- \( x \in \text{Var} \) for variables names;
- \( e \in \text{Aexpr} \) for arithmetic expressions;
- \( b \in \text{Bexpr} \) for Boolean expressions;
- \( S \in \text{Statm} \) for statements.

**Syntax:**

\[
S ::= x := e \mid \text{skip} \mid S; S \mid \text{if } b \text{ then } S \text{ else } S \mid \text{while } b \text{ do } S.
\]
State space

A basic concept in coalgebraic approach is a state specified by the **signature**: 

\[
\Sigma_{\text{State}} = \begin{align*}
\text{types} : & \quad \text{State, Var, Value} \\
\text{ops} : & \quad \text{init} : \to \text{State} \\
& \quad \text{get} : \text{Var, State} \to \text{Value} \\
& \quad \text{next} : \text{Statm, State} \to \text{State}
\end{align*}
\]

**Representation:**

- we assign to the syntactic domain *Val* a set:
  \[
  \text{Value} = \mathbb{Z} \cup \{\bot\};
  \]

- we assign to the type *Var* a countable set *Var* of variable names;

- our representation of an element of type *State* has to express a variable name together with its value:
  \[
  s : \text{Var} \to \text{Value};
  \]
  where
  \[
  s = (x, v_1), \ldots, (z, v_n)
  \]

- special states are the initial state \(s_0 = [\text{init}]\) and undefined state \(s_\bot = (\bot, \bot)\);

- state representations form the set *State* - state space.
We construct a base category $\mathcal{C}_{State}$ of states, where

- category objects are state representations from $\text{State}$; and
- category morphisms are transitions defining changes of states.

We define the representation of $next$, the transition function $[next]$:

$$\text{next} : \text{Statm} \rightarrow (\text{State} \rightarrow \text{State}),$$

that returns for a statement $S$

$$[\text{next}][S] : \text{State} \rightarrow \text{State}$$

the next state obtained from the execution of the first step of a statement $S$.

To be $\mathcal{C}_{State}$ a category we require that every infinite path (composition of morphisms) has a colimit.
Transition function

Transition function is defined for statements of the language $J$ane as follows:

$$
[next][S](s) = \begin{cases} 
  s' = s \left[ x \mapsto \llbracket e \rrbracket s \right] & \text{if } S = x := e; \\
  s & \text{if } S = \text{skip} \\
  [next][S_1; S_2](s') & \text{if } S = S_1; S_2 \text{ and } \langle S_1; S_2, s \rangle \Rightarrow \langle S_1'; S_2, s' \rangle; \\
  [next][S_2](s') & \text{if } S = S_1; S_2 \text{ and } \langle S_1; S_2, s \rangle \Rightarrow \langle S_2, s' \rangle; \\
  [next][S_1](s) & \text{if } S = \text{if } b \text{ then } S_1 \text{ else } S_2 \text{ and } \llbracket b \rrbracket s = \text{true}; \\
  [next][S_2](s) & \text{if } S = \text{if } b \text{ then } S_1 \text{ else } S_2 \text{ and } \llbracket b \rrbracket s = \text{false}; \\
  [next][S; \text{while } b \text{ do } S](s) & \text{if } S = \text{while } b \text{ do } S \text{ and } \llbracket b \rrbracket s = \text{true}; \\
  \text{abort}(s) & \text{otherwise,}
\end{cases}
$$

where $\text{abort}$ is a unique morphism which sends any state to the undefined state $s_\perp$:

$$\text{abort} : s \rightarrow s_\perp$$
Polynomial endofunctor

Now we construct the polynomial endofunctor indicated by $\Sigma_{\text{State}}$ as

$$Q : C_{\text{State}} \to C_{\text{State}}.$$

For our purposes we define a functor

$$Q(\text{State}) = 1 + \text{State}.$$

We define this functor for objects and morphisms in $C_{\text{State}}$ as follows:

$$Q(s) = s_{\bot} + [\text{next}][S]s,$$
$$Q([\text{next}][S]) = \text{abort} + [\text{next}][S].$$
A $Q$-coalgebra, also called coalgebra of type $Q$ or $Q$-system, is a pair

$$(\text{State}, [next][S]),$$

where $\text{State}$ is a state space of the coalgebra and $[next][S]$ is the structure map of the coalgebra on $\text{State}$:

$$[next][S] : \text{State} \rightarrow Q(\text{State}).$$
Example

We consider a simple program in Jane:

\[
\begin{align*}
    z &:= 0; \\
    \text{while } (y \leq x) \text{ do } (z := z + 1; x := x - y);
\end{align*}
\]

and let the initial state be \( s_0 = [x \mapsto 17, y \mapsto 5] \).

We construct over a category \( \mathcal{C}_{\text{State}} \) a polynomial endofunctor

\[ Q(\text{State}) = 1 + \text{State} \]

defined for objects and morphisms by

\[
\begin{align*}
    Q(s) &= s_\bot + [\text{next}][S]s, \\
    Q([\text{next}][S]) &= \text{abort} + [\text{next}][S].
\end{align*}
\]
Example - continuation

\[ Q(s_0) = 1 + [next][S_0]s_0 \]
\[ = [next][S_1; S_2]s_0 \]
\[ = [next][S_2]s_1 \]
\[ = [next][z := z + 1; x := x - y; S_2]s_1 \]
\[ = [next][x := x - y; S_2]s_2 \]
\[ = [next][S_2]s_3 \]
\[ = [next][z := z + 1; x := x - y; S_2]s_3 \]
\[ = [next][x := x - y; S_2]s_4 \]
\[ = [next][S_2]s_5 \]
\[ = [next][z := z + 1; x := x - y; S_2]s_5 \]
\[ = [next][x := x - y; S_2]s_6 \]
\[ = [next][S_2]s_7 \]
\[ = s_7 \]

\[ s_0 = \langle (x, 17), (y, 5) \rangle \]
\[ s_1 = \langle (x, 17), (y, 5), (z, 0) \rangle \quad [y \leq x]s_1 = \text{true} \]
\[ s_2 = \langle (x, 17), (y, 5), (z, 1) \rangle \]
\[ s_3 = \langle (x, 12), (y, 5), (z, 1) \rangle \quad [y \leq x]s_3 = \text{true} \]
\[ s_4 = \langle (x, 12), (y, 5), (z, 2) \rangle \]
\[ s_5 = \langle (x, 7), (y, 5), (z, 2) \rangle \quad [y \leq x]s_5 = \text{true} \]
\[ s_6 = \langle (x, 7), (y, 5), (z, 3) \rangle \]
\[ s_7 = \langle (x, 2), (y, 5), (z, 3) \rangle \quad [y \leq x]s_7 = \text{false} \]
Basic concepts in object oriented programming are:

- **classes:**
  - class specification is like a signature specifying methods;
  - it determines also an interface to a program;
  - for implementation of methods some constraints (assertions) are given;
  - the essentials are in a class specification;
  - the particulars are in a class implementation;

- **objects:**
  - deal with specific tasks;
  - coordination and communication is realized via sending of messages;
  - objects have private data accessible only by methods;
  - objects are grouped into classes;
  - have local states accessible by the object methods;
  - combine data structure with behaviour.
Class and object

Class

Object
Coalgebra for OOP

A class specification is a named structure consisting of a tuple of methods of the form:

\[ m_i : X \times A_i \to B_i + C_i \times X, \text{ for } i = 1, \ldots, n, \]

where

- \( X \) is a state space specification;
- \( A_i \) are inputs;
- \( B_i \) and \( C_i \) are outputs of a method \( m_i \).

A polynomial endofunctor has then a form:

\[ Q(X) = \prod_{i=1}^{n} (B_i + C_i \times X)^{A_i}. \]

- If \( C_i = \emptyset \), the associated method yields observable element from \( B_i \), but does not change a local state;
- if \( C_i \neq \emptyset \), the associated method changes a local state.

Let \( \text{State} \) be an interpretation of objects local states, with elements \( o \in \text{State} \).

A coalgebra

\[ \mathbf{m} = \langle m_1, \ldots, m_n \rangle \]

is defined by:

\[ \mathbf{m} : \text{State} \to Q(\text{State}), \]
Example

Consider a class for bank accounts with the methods:

\[
\begin{align*}
\text{balance} : & \quad X \to \mathbb{R} \\
\text{change} : & \quad X \times \mathbb{R} \to X,
\end{align*}
\]

with the assertion:

\[s.\text{change}(a).\text{balance} = s.\text{balance} + a\]

for \(s \in X\) and \(a \in \mathbb{R}\).

We interpret state space \(X\) as the set of finite sequences of reals \(\mathbb{R}^*\), i.e. each element (object of this class) \(o \in \mathbb{R}^*\) is an account of the form

\[o = \langle a_0, a_1, \ldots, a_n \rangle.\]

The polynomial endofunctor is

\[Q(\mathbb{R}^*) = \mathbb{R} \times (\mathbb{R}^*)^\mathbb{R}\]

The methods \textit{balance} and \textit{change} together form a coalgebra

\[(\text{balance}, \text{change}) : \mathbb{R}^* \to Q(\mathbb{R}^*).\]

Then the methods for an element \(o \in \mathbb{R}^*\) are

\[o.\text{balance} = a_0 + a_1 + \cdots + a_n \quad \text{and} \quad o.\text{change}(a) = \langle a_0, a_1, \ldots, a_n, a \rangle.\]
Example -continuation

The assertion

\[ o\.change(a)\.balance = o\.balance + a \]

is always valid.

The empty account is denoted by the empty sequence \( \langle \rangle \).

Let now a state be the sequence

\[ o = \langle 3.2, 5.3, -1.4 \rangle. \]

Then

\[ o\.balance = 3.2 + 5.3 - 1.4 = 7.1 \quad \text{and} \]
\[ o\.change(8.7) = \langle 3.2, 5.3, -1.4, 8.7 \rangle. \]
Example - continuation

We can consider another interpretation, which

- keeps a record of changes;
- makes additions immediately.

The interpretation of a state space $X$ is a set $\mathbb{R}^+$ of non empty sequences of reals. For an element (object)

$$o' = (a_1, \ldots, a_n) \in \mathbb{R}^+$$

the methods are

$$o'.balance = a_n \quad \text{and} \quad o'.change(a) = (a_1, \ldots, a_n, a_n + a).$$

The initial state is now $(0.0)$.

A coalgebra is

$$(balance, change) : \mathbb{R}^+ \to Q(\mathbb{R}^+)$$

and the methods also satisfy the assertion

$$o'.change(a).balance = o'.balance + a.$$
Component based programming

Component based programming is about
- how to create an application program from prefabricated components together;
- with new software providing both glue between the components and new functionality.

A component
- is an independent deployable entity;
- it interacts with the environment by typed ports specified in its interface;
- it has no observable state, its initial state is established after its deployment.
- can be generic, substitution of its parameters by appropriate arguments (of proper types) enable its using for different purposes.

The typed ports
- serve as end points interactions;
- they enable transfer of data of some type in required direction;
- cooperation between components can be performed only through ports of corresponding types.
From components to an application

⇓ composition
Coalgebra for components

To define coalgebras for components we denote by

- $I$ a set of typed input ports;
- $O$ a set of typed output ports.

Then an interface of a component is a pair

$$(I, O)$$

and a component is an arrow

$$comp : I \rightarrow O.$$ 

To ensure genericity, we use a strong monad $B$ over a base category and then a coalgebra of a component is

$$c_{comp} : X_{comp} \times I_{comp} \rightarrow B(X_{comp} \times O)$$

For each state $s \in X_{comp}$ the behaviour is organized as a tree because it depends on the sequences of input values. In this tree:

- elements of $O$ are nodes;
- elements of $I$ are labels of edges.
Example

Consider a buffer $\text{Buffer}$ as a component that stores input data elements ($\text{Message}$) and returns them in responding to request. This component has one input port and one output port and operations:

\begin{align*}
\text{put} : & \rightarrow \text{Message} \times \text{Buffer} \\
\text{pick} : & \rightarrow \text{Buffer} \rightarrow \text{Message} \times \text{Buffer}
\end{align*}

Let

- $M$ be a type of messages;
- $M^*$ represents a buffer;
- $I = M + 1$ represents inputs;
- $O = 1 + M$ represents outputs, where 1 stands for nullary datatype.

The polynomial endofunctor is then

$$Q(M^*) = (M^* \times O)^I$$

and the coalgebra for this component is

$$c : M^* \rightarrow Q(M^*).$$
Thank you for your attention